

On a non linear third - order parabolic equation

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Abstract

Aim of this paper is the qualitative analysis of the solution of a boundary value problem for a third-order non linear parabolic equation which describes several dissipative models. When the source term is linear, the problem is explicitly solved by means of a Fourier series with properties of rapid convergence. In the non linear case, appropriate estimates of this series allow to deduce the asymptotic behaviour of the solution.

1 Introduction

We refer to the non linear equation

$$(1.1) \quad \mathcal{L}_\varepsilon u = F(x, t, u, u_x, u_t)$$

where \mathcal{L}_ε is the third - order parabolic operator:

$$(1.2) \quad \mathcal{L}_\varepsilon = \partial_{xx}(\varepsilon \partial_t + c^2) - \partial_t(\partial_t + a).$$

In (1.1)-(1.2) ε, a, c are positive constants and $F = F(x, t, u, u_x, u_t)$ is a prefixed function.

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The equation (1.1) characterizes the evolution of several dissipative models such as the motions of viscoelastic fluids or solids, the sound propagation in viscous gases, the heat conduction at low temperature and some Josephson effects in superconductivity when dissipative causes are not negligible (for details see, e.g, [6][7]).

The analysis of several initial - boundary problems related to the equation (1.1) has been discussed in many papers (see, for example [10] and its references).

In [6], the fundamental solution $K(x, t)$ of the operator \mathcal{L}_ε has been determined explicitly by means of convolutions of Bessel function, and so, when $F = f(x, t)$ is linear, the initial value problem has been discussed. Further, the linear strip problem has been solved in [8] by means of a Green function G connected with $K(x, t)$. However, it seems difficult to deduce an exhaustive asymptotic analysis by means of this solution.

In this paper, both linear and non linear cases are analyzed. The Green function of the linear strip problem is determined by Fourier series which is characterized by properties of rapid convergence and allows to establish an exponential decrease of the solution. Moreover, the non - linear problem is reduced to an integral equation with kernel G and, by means of suitable properties of G , the asymptotic analysis of the solutions is achieved.

2 Some typical problems in superconductivity

As an example of physical models related to the equation (1.1), we will consider a typical case of non linear phenomenon concerning the Josephson effects in Superconductivity.

Let $\varphi = \varphi(x, t)$ be the phase difference of the wave functions related to the two superconductors, and let γ be the normalized current bias; when $F = \sin\varphi - \gamma$, the equation (1.1) gives the perturbed Sine- Gordon equation [1]:

$$(2.1) \quad \varepsilon\varphi_{xxt} + \varphi_{xx} - \varphi_{tt} - a\varphi_t = \sin\varphi - \gamma.$$

The terms $\varepsilon\varphi_{xxt}$ and $a\varphi_t$ characterize the dissipative normal electron current flow along and across the junction, respectively, and they represent

the *perturbations* with respect to the classic Sine Gordon equation [11, 14, 16]. When the surface resistance is negligible [4, 12, 13], then ε is vanishing and a singular perturbation problem for the equation (1.1) could be appear. As for the coefficient a of (2.1), it depends on the shunt conductance [5] and generally one has $a \ll 1$ [3, 13, 14]. However, if the resistance of the junction is so low to short completely the capacitance, the case $a \gg 1$ arises [1, 17, 18].

If l is the normalized length of junction, a first boundary problem refers to the phase - boundary specifications for $x = 0$ and $x = l$ [5]. Moreover, according to the junction geometry, other typical boundary value problems could be considered. For instance, in an overlap geometry [1, 14], if η is the normalized measure of the external magnetic field applied in the plane of the junction [4], it has:

$$(2.2) \quad \varphi_x(0, t) = \varphi_x(l, t) = \eta.$$

Instead, if an 'open-circuit' is examined, then boundary conditions (2.2) assume the homogeneous form [3], while, for the electrodynamics of a Josephson junction in two spatial dimensions, other conditions on $(0, l)$ can be considered [15].

As an example, we will analyze the problem related to the phase boundary specifications but the analysis can be applied to the other problems.

3 The linear strip problem

Let l, T be arbitrary positive constants and let

$$\Omega = \{(x, t) : 0 < x < l, \quad 0 < t \leq T\}.$$

In the linear case, the strip problem is:

$$(3.1) \quad \begin{cases} \partial_{xx}(\varepsilon u_t + c^2 u) - \partial_t(u_t + au) = f(x, t), & (x, t) \in \Omega, \\ u(x, 0) = g_0(x), \quad u_t(x, 0) = g_1(x), & x \in [0, l], \\ u(0, t) = 0, \quad u(l, t) = 0, & 0 < t \leq T. \end{cases}$$

As it's well known, the problem (3.1) can be solved explicitly by means of the Fourier method, as soon as the Green function for the operator (3.1)₁ has been determined. For this let :

$$(3.2) \quad \gamma_n = \frac{n\pi}{l}, \quad b_n = c\gamma_n, \quad h_n = \frac{1}{2}(a + \varepsilon\gamma_n^2), \quad \omega_n = \sqrt{h_n^2 - b_n^2};$$

and

$$(3.3) \quad H_n(t) = \frac{1}{\omega_n} e^{-h_n t} \sinh(\omega_n t).$$

By standard techniques, the Green function related to (3.1) can be given the form:

$$(3.4) \quad G(x, t, \xi) = \frac{2}{l} \sum_{n=1}^{\infty} H_n(t) \sin \gamma_n \xi \sin \gamma_n x.$$

whose properties will be discussed in the next section 4.

Then, if we consider the functions:

$$(3.5) \quad u_{g_i} = \int_0^l g_i(\xi) G(x, \xi, t) d\xi \quad i = 0, 1$$

and

$$(3.6) \quad u_f = G * f = \int_0^t d\tau \int_0^l f(\xi, \tau) G(x, \xi, t - \tau) d\xi,$$

the formal solution of (3.1) is given by:

$$(3.7) \quad u(x, t) = u_{g_1} + (\partial_t + a - \varepsilon \partial_{xx}) u_{g_0} - u_f(x, t).$$

4 Properties of the G function

At first we will analyze the convergence and the asymptotic properties of the series (3.4) which defines G . We will prove that the order of the terms of G is at least n^{-2} and the function G is exponentially vanishing as $t \rightarrow \infty$. For this we put:

$$(4.1) \quad p = \frac{c^2}{\varepsilon + a(l/\pi)^2}, \quad q = \frac{a + \varepsilon(\pi/l)^2}{2}, \quad \beta \equiv \min(p, q),$$

and we will prove the following lemma:

Lemma 4.1- *Whatever the constants a, ε, c^2 may be in \mathbb{R}^+ , the function $G(x, \xi, t)$ defined in (3.4) and all its time derivatives are continuous functions in Ω . Moreover, everywhere in Ω , it results:*

$$(4.2) \quad |G(x, \xi, t)| \leq M e^{-\beta t}, \quad \left| \frac{\partial^j G}{\partial t^j} \right| \leq N_j e^{-\beta t}, \quad j \in \mathbb{N}$$

where M, N_j are constants depending on a, ε, c^2 .

Proof. When $c^2 = a\varepsilon$, the operator L_ε can be reduced to the wave operator. If $c^2 < a\varepsilon$, for all $n \geq 1$, one has $h_n > b_n$. So, if k is an arbitrary constant such that $c^2/a\varepsilon < k < 1$, the Cauchy inequality assures that:

$$(4.3) \quad \frac{b_n}{\sqrt{k}} \leq \frac{1}{2} (\varepsilon \gamma_n^2 + a) = h_n.$$

Then, if $X_n = (b_n/h_n)^2$, one has:

$$(4.4) \quad (1 - k)^{\frac{1}{2}} \leq \frac{\omega_n}{h_n} = (1 - X_n)^{\frac{1}{2}} < 1 - X_n/2.$$

Further, for all $n \geq 1$, it results:

$$(4.5) \quad \frac{b_n^2}{2h_n} \geq p, \quad h_n > \frac{\varepsilon\pi^2}{2l^2} n^2 = (q - a/2) n^2$$

and hence, by (4.4) and (4.5)₁, it follows:

$$(4.6) \quad e^{-t(h_n - \omega_n)} \leq e^{-pt}.$$

Moreover, the estimates (4.4), (4.5)₂, (4.6) imply that the terms H_n defined by (3.3) verify:

$$(4.7) \quad |H_n| \leq \frac{(1-k)^{-1/2}}{q - a/2} \frac{e^{-pt}}{n^2}, \quad \forall n \geq 1$$

and hence, (4.2)₁ follows. As for (4.2)₂, we observe:

$$(4.8) \quad h_n - \omega_n = \frac{b_n^2}{h_n + \omega_n} \leq \frac{2c^2}{\varepsilon}, \quad \forall n \geq 1$$

and by means of standard computations one obtains (4.2)₂.

Consider now the case $c^2 > a\varepsilon$, and let

$$(4.9) \quad N_{1,2} = \frac{cl}{\varepsilon\pi} [1 \mp (1 - a\varepsilon/c^2)^{1/2}].$$

If k is a positive constant less than one, let $N_{1,2}^*$, N_k be the lowest integers such that

$$(4.10) \quad N_1^* < N_1, \quad N_2^* > N_2; \quad , \quad N_k > \frac{cl}{\varepsilon\pi\sqrt{k}} [1 + (1 - a\varepsilon k/c^2)^{1/2}].$$

It suffices to analyse only the hyperbolic terms H_n with $n \geq N_2^*$. Similarly to (4.4) for all $n \geq N_k$, it results :

$$(4.11) \quad \omega_n \geq h_n (1 - k)^{1/2}.$$

So (4.5)₂, (4.6), (4.8) and (4.11) imply the estimates (4.1) also when $c^2 > a\varepsilon$.

Obviously, the lemma holds also when $N_{1,2}$ are integers. In this case, the constants M and N_j could depend on t . \square

As it is known, as for the x -differentiation of Fourier's series like (3.4), attention must be paid to convergence problems. For this, we will consider x -derivatives of the operator $(\varepsilon\partial_t + c^2)G$ instead of G and G_t .

Lemma 4.2. - *Whatever a, ε, c^2 may be, the function $G(x, \xi, t)$ defined in (3.4) is such that:*

$$(4.12) \quad |\varepsilon G_t + c^2 G| \leq A_0 e^{-\beta t},$$

$$(4.13) \quad |\partial_{xx}(\varepsilon G_t + c^2 G)| \leq A_1 e^{-\beta t},$$

where A_0 and A_1 are constants depending on a, ε, c^2 .

Proof. As for the hyperbolic terms in H_n , we consider $X_n = (\frac{b_n}{h_n})^2 < 1$ and put:

$$(4.14) \quad \varphi_n = h_n(-1 + \sqrt{1 - X_n}).$$

It results:

$$(4.15) \quad \varepsilon \dot{H}_n + c^2 H_n = \frac{1}{2\omega_n} e^{-h_n t} \{ (c^2 + \varepsilon \varphi_n) e^{\omega_n t} - [c^2 - \varepsilon(h_n + \omega_n)] e^{-\omega_n t} \},$$

where Taylor's formula assures that:

$$(4.16) \quad c^2 + \varepsilon \varphi_n = \frac{c^2 a}{a + \varepsilon \gamma_n^2} - \varepsilon h_n X_n^2 - \frac{3}{16} \int_0^{X_n} \frac{(X_n - y)^2}{(1 - y)^{5/2}} dy.$$

Further, it is possible to prove that, for all n , it is:

$$(4.17) \quad X_n < \frac{\alpha^2}{n^2} \quad (\alpha = 2cl/\varepsilon\pi)$$

while, for every $h > 0$ and $n \geq \alpha(1+h)$, one has:

$$(4.18) \quad \int_0^{X_n} \frac{(X_n - y)^2}{(1 - y)^{5/2}} dy \leq \frac{2}{3} X_n^2 \{(h+1)^3/[h(h+2)]^{3/2} - 1\}.$$

So, estimates (4.17) and (4.18) allow to obtain:

$$(4.19) \quad |c^2 + \varepsilon \varphi_n| \leq \frac{1}{n^2} (5\alpha^2/4 + \alpha_1/8),$$

where $\alpha_1 = \alpha^4[(h+1)^3/(h^2+2h)^{3/2} - 1]$.

The estimate of Lemma 4.1 together with (4.19) show that the terms of the serie related to the operator (4.12) have order at least of n^{-4} . So, it can be differentiated term by term with respect to x and the estimate (4.13) can be deduced. \square

As solution of the equation $\mathcal{L}_\varepsilon v = 0$ we will mean a continuous function $v(x, t)$ which has continuous the derivatives $v_t, v_{tt}, \partial_{xx}(\varepsilon v_t + c^2 v)$ and these derivatives verify the equation.

So, we are able to prove the following theorem:

Theorem 4.1- *The function $G(x, t)$ defined in (3.4) is a solution of the equation*

$$(4.20) \quad \mathcal{L}_\varepsilon G = \partial_{xx}(\varepsilon G_t + c^2 G) - \partial_t(G_t + aG) = 0.$$

Proof. The uniformly convergence proved in Lemma 4.1 and Lemma 4.2 allow to deduce that:

$$(4.21) \quad (\partial_t + a)G_t = \frac{2}{l} \sum_{n=1}^{\infty} \{[-b_n^2 + (2h_n - a)h_n]H_n + (a - 2h_n)e^{-h_n t} \cosh \omega_n t\} \sin \gamma_n \xi \sin \gamma_n x,$$

and

$$(4.22) \quad \partial_{xx}(\varepsilon\partial_t + c^2)G = \frac{2}{l} \sum_{n=1}^{\infty} [(\varepsilon\gamma_n^2 h_n - b_n^2)H_n - \varepsilon\gamma_n^2 e^{-h_n t} \cosh \omega_n t] \sin \gamma_n \xi \sin \gamma_n x,$$

and hence, (4.20) follows. \square

5 Properties of the convolution

To obtain the explicit solution of the linear strip problem, the properties of the convolution of function G with the data must be analyzed. For this, let $g(x)$ be a continuous function on $(0, l)$ and let consider:

$$(5.1) \quad u_g(x, t) = \int_0^l g(\xi) G(x, \xi, t) d\xi.$$

$$(5.2) \quad u_g^*(x, t) = (\partial_t + a - \varepsilon\partial_{xx})u_g(x, t).$$

So the following lemma holds:

Lemma 5.1- *If $g(x)$ is a $C^1(0, l)$, then the function u_g defined by (5.1) is a solution (n.4) of the equation $\mathcal{L}_\varepsilon = 0$ such that:*

$$(5.3) \quad \lim_{t \rightarrow 0} u_g(x, t) = 0 \quad \lim_{t \rightarrow 0} \partial_t u_g(x, t) = g(x),$$

uniformly for all $x \in (0, l)$.

Proof. Lemma (4.1)-(4.2) and continuity of g assure that function (5.1) and the partial derivatives required by the solution converge absolutely for all $(x, t) \in \Omega$. Hence, by theorem 4.1, one deduces:

$$(5.4) \quad \partial_{xx}(\varepsilon\partial_t + c^2)u_g = \int_0^l g(\xi) \partial_{xx}(\varepsilon\partial_t + c^2)G(x, \xi, t) d\xi = \partial_t(\partial_t + a)u_g.$$

Besides, the hypotheses on the function g and (4.2), imply (5.3)₁.

Moreover, one has:

$$(5.5) \quad G_t = -\frac{2}{\pi} \frac{\partial}{\partial \xi} \sum_{n=1}^{\infty} \dot{H}_n(t) \frac{\cos \gamma_n \xi}{n} \sin \gamma_n x,$$

and hence:

$$(5.6) \quad \begin{aligned} \partial_t u_g = & -\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\dot{H}_n(t)}{n} [g(\xi) \cos \gamma_n \xi]_0^l \sin \gamma_n x + \\ & + \frac{2}{\pi} \int_0^l \sum_{n=1}^{\infty} \dot{H}_n(t) g'(\xi) \frac{\cos \gamma_n \xi}{n} \sin \gamma_n x d\xi. \end{aligned}$$

So, if $\eta(x)$ is the Heaviside function, from (5.6), one obtains:

$$(5.7) \quad \lim_{t \rightarrow 0} \partial_t u_g = \frac{x}{l} [g(l) - g(0)] + g(0) - \int_0^l g'(\xi) [\eta(\xi - x) + \frac{x}{l} - 1] d\xi = g(x). \square$$

Lemma 5.2- *If $g(x) \in C^3(0, l)$ with $g^{(i)}(0) = g^{(i)}(l) = 0$ ($i = 1, 2, 3$), then the function u_g^* defined in (5.2) is a solution (n.4) of the equation $\mathcal{L}_\varepsilon = 0$ such that:*

$$(5.8) \quad \lim_{t \rightarrow 0} u_g^*(x, t) = g \quad \lim_{t \rightarrow 0} \partial_t u_g^*(x, t) = 0$$

uniformly for all $x \in (0, l)$.

Proof. The hypotheses on $g(x)$ allow to have

$$(5.9) \quad \partial_{xx} u_g(x, t) = \int_0^l g''(\xi) G(x, \xi, t) d\xi = u_{g''}(x, t).$$

So, by Lemma 5.1 and expression (5.9), one obtains :

$$(5.10) \quad \partial_t(\partial_t + a)u_g^* = \partial_t(\partial_t + a)[(\partial_t + a)u_g - \varepsilon u_{g''}] = \partial_{xx}(\varepsilon \partial_t + c^2)u_g^*.$$

Moreover, it results:

$$(5.11) \quad \partial_t u_g^* = \partial_t^2 u_g + a \partial_t u_g - \varepsilon \partial_t u_{g''} = c^2 u_{g''},$$

and hence, (5.3)₁ implies (5.8)₂.

Finally, owing to (5.3)-(5.9), one has:

$$(5.12) \quad \lim_{t \rightarrow 0} u_g^* = \lim_{t \rightarrow 0} [(\partial_t + a)u_g - \varepsilon u_{g''}] = g(x). \square$$

6 Explicit solution of the strip linear problem

Let us consider the homogeneous case. From Lemma 5.1 and Lemma 5.2 the following result is obtained:

Theorem 6.1- *If the initial data $g_1(x)$, and $g_0(x)$ verify the hypotheses of Lemma 5.1 and Lemma 5.2, then the function:*

$$(6.1) \quad u(x, t) = u_{g_1} + (\partial_t + a - \varepsilon \partial_{xx})u_{g_0}$$

represents a solution (n.4) of the homogeneous strip problem:

$$(6.2) \quad \begin{cases} \partial_{xx}(\varepsilon u_t + c^2 u) - \partial_t(u_t + au) = 0, & (x, t) \in \Omega, \\ u(x, 0) = g_0(x), \quad u_t(x, 0) = g_1(x), & x \in [0, l], \\ u(0, t) = 0, \quad u(l, t) = 0, & t \in (0, T]. \end{cases}$$

As for the non-homogeneous problem, let:

$$(6.3) \quad u_f(x, t) = \int_0^t d\tau \int_0^l f(\xi, \tau) G(x, \xi, t - \tau) d\xi,$$

the following theorem holds:

Theorem 6.2- *If the function $f(x, t)$ is a continuous function in Ω with continuous derivative with respect to x , then the function u_f represents a solution (n.4) of the non-homogeneous strip problem:*

$$(6.4) \quad \begin{cases} \partial_{xx}(\varepsilon \partial_t + c^2)u - \partial_t(\partial_t + a)u = -f(x, t), & (x, t) \in \Omega, \\ u(x, 0) = 0, \quad \partial_t u(x, 0) = 0, & x \in [0, l], \\ u(0, t) = 0, \quad u(l, t) = 0, & t \in (0, T]. \end{cases}$$

Proof. By (3.4) one has:

$$(6.5) \quad \lim_{\tau \rightarrow t} u_f(x, t) = 0,$$

and hence:

$$(6.6) \quad \partial_t u_f(x, t) = \int_0^t d\tau \int_0^l f(\xi, \tau) G_t(x, \xi, t - \tau) d\xi.$$

Moreover, as in (5.7), it is possible to prove that:

$$(6.7) \quad \lim_{\tau \rightarrow t} \int_0^l f(\xi, \tau) G_t(x, \xi, t - \tau) d\xi = f(x, t),$$

and one obtains:

$$(6.8) \quad \partial_t^2 u_f = f(x, t) + \int_0^t d\tau \int_0^l f(\xi, \tau) G_{tt}(x, \xi, t - \tau) d\xi.$$

So that, Lemma 4.2, Theorem 4.1 and (6.4)-(6.6) assure that:

$$\partial_{xx}(\varepsilon\partial_t + c^2)u_f = \int_0^t d\tau \int_0^l f(\xi, \tau) \partial_{xx}(\varepsilon G_t + c^2 G) d\xi = \partial_t(\partial_t + a)u_f - f(x, t). \quad (6.9)$$

Finally, owing to (6.3)-(6.6) and Lemma 4.1, if B_i ($i=1,2$) are two positive constants, one has:

$$(6.10) \quad |u_f| \leq B_1(1 - e^{-\beta t}); \quad |\partial_t u_f| \leq B_2(1 - e^{-\beta t}),$$

from which (6.4)₂ follow. \square

Moreover, the uniqueness is a consequence of the energy-method (see, for example, [9]) and we have:

Theorem 6.3- *When the source term $f(x, t)$ satisfies theorem 6.2, and the initial data (g_0, g_1) satisfy theorem 6.1, then the function*

$$(6.11) \quad u(x, t) = u_{g_1} + (\partial_t + a - \varepsilon \partial_{xx})u_{g_0} - u_f$$

is the unique solution (n.4) of the linear non-homogeneous strip problem (3.1) \square

7 Asymptotic properties

At first we will consider the linear problem (3.1). Obviously, the asymptotic properties of the solution depend on the behaviour of the source term $f = f(x, t)$. For instance, one has:

Theorem 7.1- *When the source term $f(x, t)$ satisfies the condition:*

$$(7.1) \quad |f(x, t)| \leq h \frac{1}{(k + t)^{1+\alpha}} \quad (h, k, \alpha = \text{const} > 0),$$

then the solution of (3.1) is vanishing as $t \rightarrow \infty$, at least as $t^{-\alpha}$.

Moreover, when

$$(7.2) \quad |f(x, t)| \leq C e^{-\delta t} \quad (C, \delta = \text{const} > 0),$$

one has:

$$(7.3) \quad |u(x, t)| \leq k e^{-\delta^* t} \quad \delta^* = \min\{\beta, \delta\}, \quad k = \text{const}.$$

Proof. Referring to the explicit solution (6.11) of the problem (3.1), we observe that, owing to the Lemma 4.1, the terms u_{g_0} and u_{g_1} defined by (5.1) are exponentially vanishing as $t \rightarrow \infty$. Moreover, the convolution u_f given by (3.6) can be estimated by (4.2) too and the hypotheses on $f(x, t)$ imply the statement. \square

Consider now the non linear problem:

$$(7.4) \quad \begin{cases} \varepsilon u_{xxt} + c^2 u_{xx} - u_{tt} - a u_t = F(x, t, u), \\ u(x, 0) = g_0, \quad u_t(x, 0) = g_1, \\ u(0, t) = 0, \quad u(l, t) = 0. \end{cases}$$

When one applies the results of section 6 (see 6.3), one obtains the following integral equation:

$$(7.5) \quad u(x, t) = \int_0^l g_1(\xi) G(x, \xi, t) d\xi + (\partial_t + a - \varepsilon \partial_{xx}) \int_0^l g_0(\xi) G(x, \xi, t) d\xi + \\ \int_0^t d\tau \int_0^l G(x, \xi, t - \tau) F(\xi, \tau, u(\xi, \tau)) d\xi,$$

to which the previous estimates can be applied.

For instance, as for the perturbed Sine Gordon equation (2.1), one has:

$$(7.6) \quad |F(x, t, u)| = |\sin u + \gamma| \leq \gamma_1, \quad \gamma_1 > 0$$

and so the Lemma 4.1 implies that the solution of the related strip problem is bounded for all t .

Another example of applications could be given by exponentially vanishing non linear source $F(x,t,u(x,t))$ such that (see [2]):

$$(7.7) \quad |F(x,t,u(x,t))| \leq \text{const. } e^{-\mu t} \quad (\mu > 0)$$

As consequence of (7.5) and (4.2), the solution of (7.4) will vanish exponentially, too.

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Acknowledgement. This research was supported by Italian Ministry of University and Scientific Research.